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AXISYMMETRIC STRESS DISTRIBUTION IN THE VICINITY OF AN EXTERNAL CRACK UNDER GENERAL SURFACE LOADINGS

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Abstract—A solution is derived of the equations of equilibrium appropriate to the axisymmetric loading on the faces of a plane crack covering the outside of a circle of radius a in an infinite isotropic elastic body. Abel transforms of stress and displacement components at an arbitrary point of the solid are known in the literature in terms of the Abel transforms of the jumps of the stress and displacement components at the crack plane. Limiting values of these expressions, as we approach the crack plane from either side, are a great help in dealing with the problem. The boundary conditions lead to Abel type integral equations which admit closed form solutions. The expressions for stress and displacement components on the crack plane are obtained explicitly in terms of the prescribed stress components on the crack surfaces. The surface tractions on the crack are symmetrical about the centre of the circle but not necessarily self-equilibrating.

In order to illustrate the use of general formulae, some special cases of the loading functions are discussed in detail. In the first case the upper crack surface is subjected to a uniform normal stress acting over a circular ring of inner and outer radii ε and c, respectively, while the lower crack surface is free from tractions. In a second case the upper crack surface is subjected to a radially decaying shear load acting over a circular ring of inner and outer radii ε and c, respectively, while the lower crack surface is free from tractions. Also, we obtain the results for normal and shear concentrated ring loads by taking the limit as $\varepsilon \to c$ while the total applied load is kept fixed. The expressions for displacement components on the crack plane are in terms of complete as well as incomplete elliptic integrals of the first and second kinds. Numerical calculations for the normal components of displacement on the crack surfaces are carried out and the results are presented graphically.

NOMENCLATURE

(r, θ, z)	cylindrical polar coordinates
$u_r^{(1)}, u_z^{(1)}$	displacement components for the upper half space ($z > 0$)
$u_r^{(2)}, u_z^{(2)}$	displacement components for the lower half space ($z < 0$)
$\sigma_{rz}^{(k)}, \sigma_{zz}^{(k)}, \sigma_{rr}^{(k)}, \sigma_{\theta\theta}^{(k)}$,	corresponding stress components with $k = 1, 2$
$u_z^{(1)}(r, 0)$	limiting value of $u_z^{(1)}(r, z)$ as $z \to 0^+$
$u_z^{(2)}(r, 0)$	limiting value of $u_z^{(2)}(r, z)$ as $z \to 0^-$
λ, μ	Lamé's constants
ν	Poisson's ratio
$\hat{E}$	Young's modulus
$K_{\rm I}, K_{\rm II}$	mode I and mode II stress intensity factors
$K(k), E(k)$	complete elliptic integrals
$F(\phi, k), E(\phi, k)$	incomplete elliptic integrals
max (r, a)	greater of the two numbers r and a
min (r, a)	smaller of the two numbers r and a
sgn (x)	signum function
$H_0(x)$	heaviside unit function.

1. INTRODUCTION

The problem of determining the distribution of stress in an infinite isotropic elastic solid, when pressure is applied to the faces of a flat external crack covering the outside of a circle, has been studied by Uflyand (1959). Uflyand expresses the Boussinesq–Papkovich solution of the equations of elastic equilibrium in toroidal coordinates and then uses the Mehler–Fock transform to solve the relevant boundary value problem. Lowengrub and Sneddon (1965) use cylindrical coordinates (r, θ, z) and assume: (i) that the crack face $z = 0^+, r \geqslant 1$ is loaded exactly the same way as the face $z = 0^-, r \geqslant 1$; and (ii) that the loading function $p(r, \theta)$ is an even function of θ. The restriction (ii) is not in any sense limiting; any problem in which it is not satisfied can be solved in exactly the same way except that the Fourier

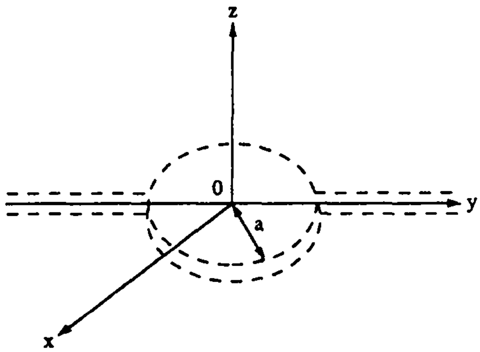

Fig. 1. Problem geometry.

series representing $p(r, \theta)$ will then contain sine terms in addition to cosine terms. In the method used in Lowengrub and Sneddon (1965) the condition (i) cannot be relaxed; it must be retained in order that the problem can be reduced to a mixed boundary value problem for a half space. It should be emphasized that Uflyand's solution does not involve making this assumption of symmetry about the plane of the crack. The methods of Uflyand (1959) and Lowengrub and Sneddon (1965) are discussed in some detail in a book by Sneddon and Lowengrub (1969). A more recent account of the developments may be found in a book by Kassir and Sih (1975).

In the present paper we use cylindrical coordinates (r, θ, z) and restrict our attention to the case in which the loading on the crack surface is axisymmetric. In this special case we relax the condition (i) of Lowengrub and Sneddon (1965). That is, we consider the problem of determining the axisymmetric distribution of stress in an infinite isotropic elastic solid containing a flat external circular crack subjected to surface tractions which are not necessarily self-equilibrating (see Fig. 1). The faces of the crack are described by the relations $z = 0^+$ with $r \geqslant a$ and $z = 0^-$ with $r \geqslant a$ (where $a > 0$). To solve the problem we use the expressions for stress and displacement components for the upper half space ($z > 0$) and the lower half space ($z < 0$) in terms of the displacement and stress discontinuities at the plane $z = 0$. These expressions together with their limiting values as $z \to 0^+$ and as $z \to 0^-$ have been given by the authors recently in Parihar and Rao (1991). The limiting values of stress and displacement components together with the boundary conditions on the crack faces and the continuity conditions on the rest of the plane $z = 0$ ($0 \leqslant r < a$) lead to Abel integral equations which admit closed form solutions. Thus, we are able to determine the displacement and stress discontinuities in terms of the prescribed surface transactions on the crack faces.

In order to illustrate the use of the formulae obtained for the general loading some special cases are discussed in detail. In the first case the solid is stretched in the positive z-direction on the upper crack surface and it is stretched in the negative z-direction on the lower crack surface. The stretching is uniform and occurs over a circular ring of inner and outer radii ε and c, respectively, where $a < \varepsilon < c < \infty$. The special case of concentrated ring load normal to the crack surfaces is studied as the limiting case of uniform stretching ($\varepsilon \to c$). In the next special case the solid is sheared radially away from the origin on the upper crack surface and toward the origin on the lower crack surface. The shearing action is variable and occurs over a circular ring of inner and outer radii ε and c, respectively. The special case of concentrated ring load in the radial direction is studied as a limiting case of aforementioned shear load ($\varepsilon \to c$). Some graphical results are presented for the case in which the lower crack surface is free from tractions and the upper crack surface is subjected to normal surface tractions.

2. STRESS FIELD IN THE NEIGHBOURHOOD OF THE CRACK PLANE

Let the displacement components in the upper half space ($z > 0$) be denoted by $u_r^{(1)}(r, z)$, $u_z^{(1)}(r, z)$ and their limiting values as $z \to 0^+$ by $u_r^{(1)}(r, 0)$, $u_z^{(1)}(r, 0)$, respectively.

Similarly, let the displacement components in the lower half space ($z < 0$) be denoted by $u_r^{(2)}(r, z)$, $u_z^{(2)}(r, z)$ and their limiting values as $z \to 0^-$ by $u_r^{(2)}(r, 0)$, $u_z^{(2)}(r, 0)$, respectively. Using an obvious notation for the stress components we set

$$\frac{\partial}{\partial \rho} \int_0^\rho \frac{\rho[u_r^{(1)}(r,0) - u_r^{(2)}(r,0)]\,\mathrm{d}r}{\sqrt{\rho^2 - r^2}} = A(\rho), \quad \rho > 0, \tag{1}$$

$$\frac{\partial}{\partial \rho} \int_0^\rho \frac{r[u_z^{(1)}(r,0) - u_z^{(2)}(r,0)]\,\mathrm{d}r}{\sqrt{\rho^2 - r^2}} = B(\rho), \quad \rho > 0, \tag{2}$$

$$\int_0^\rho \frac{\rho[\sigma_{rz}^{(1)}(r,0) - \sigma_{rz}^{(2)}(r,0)]\,\mathrm{d}r}{\sqrt{\rho^2 - r^2}} = C(\rho), \quad \rho > 0, \tag{3}$$

$$\int_0^\rho \frac{r[\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0)]\,\mathrm{d}r}{\sqrt{\rho^2 - r^2}} = D(\rho), \quad \rho > 0, \tag{4}$$

then the Abel transforms of the stress and displacement components $\sigma_{zz}(r, z)$, $\sigma_{rz}(r, z)$, $u_r(r, z)$ and $u_z(r, z)$ are given in Parihar and Rao (1991) [see eqns (3.29)–(3.34)] in terms of the functions A, B, C, D. The limiting values of these Abel transforms of stress and displacement components as $z \to 0^+$ and as $z \to 0^-$ are given by eqns (3.35)–(3.38) in Parihar and Rao (1991) and eqns (1), (2), (4) in this section together with

$$\frac{\partial}{\partial \rho} \int_0^\rho \frac{\rho[\sigma_{rz}^{(1)}(r,0) - \sigma_{rz}^{(2)}(r,0)]\,\mathrm{d}r}{\sqrt{\rho^2 - r^2}} = \frac{\mathrm{d}}{\mathrm{d}\rho} C(\rho), \quad \rho > 0. \tag{5}$$

When integrated with respect to ρ, this equation involves an arbitrary constant which we can settle by using the condition

$$C(\infty) = 0. \tag{6}$$

This condition follows from (3) when we use a physical consideration that the resultant shear stress on the plane $z = 0$ must vanish. If we make use of the formulae (3.33)–(3.36), (5.1) and (5.2) of Parihar and Rao (1991) and the integral (A1) of Appendix A, we can show that

$$r[\sigma_{zz}^{(1)}(r,0) + \sigma_{zz}^{(2)}(r,0)] = \frac{2\mu}{\pi(\lambda + 2\mu)} \frac{\mathrm{d}}{\mathrm{d}r} \int_r^\infty \frac{[2(\lambda + \mu)B(t) + C(t)]\,t\,\mathrm{d}t}{\sqrt{t^2 - r^2}}, \quad r > 0, \tag{7}$$

$$\sigma_{rz}^{(1)}(r,0) + \sigma_{rz}^{(2)}(r,0) = \frac{2\mu}{\pi(\lambda + 2\mu)} \frac{\mathrm{d}}{\mathrm{d}r} \int_r^\infty \frac{[2(\lambda + \mu)A(t) - D(t)]\,\mathrm{d}t}{\sqrt{t^2 - r^2}}, \quad r > 0. \tag{8}$$

Also, using eqns (3.37) and (3.34) of Parihar and Rao (1991) we can write

$$\int_0^\rho \frac{[u_r^{(1)}(r,0) + u_r^{(2)}(r,0)]\,\mathrm{d}r}{\sqrt{\rho^2 - r^2}} = \frac{1}{\rho}\left[\int_0^\rho L(s)\,\mathrm{d}s + \alpha_1\right], \quad \rho > 0, \tag{9}$$

where α_1 is the arbitrary constant of integration and we have

$$L(\rho) = -\frac{\rho}{\pi\mu(\lambda + 2\mu)} \int_0^\infty \frac{[2\mu^2 B(t) - (\lambda + 3\mu)C(t)]\,\mathrm{d}t}{\rho^2 - t^2}, \quad \rho > 0. \tag{10}$$

Inverting the Abel operator in (9) and simplifying we get

$$u_r^{(1)}(r,0)+u_r^{(2)}(r,0) = \frac{2}{\pi r}\int_0^r \frac{\rho L(\rho)\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}, \quad r>0. \tag{11}$$

We note that in this process the coefficient of the term involving α_1 becomes zero. Substituting $L(\rho)$ from (10) into (11), interchanging the order of integrations and using the integral (A1) yields

$$r[u_r^{(1)}(r,0)+u_r^{(2)}(r,0)] = \Delta(r)-\Delta(0), \quad r>0, \tag{12}$$

where

$$\Delta(r) = \frac{1}{\pi\mu(\lambda+2\mu)}\int_r^\infty [2\mu^2 B(t)-(\lambda+5\mu)C(t)]\frac{t\,\mathrm{d}t}{\sqrt{t^2-r^2}}, \quad r>0. \tag{13}$$

In an analogous manner, starting with eqn (3.38) of Parihar and Rao (1991), we can show that

$$u_z^{(1)}(r,0)+u_z^{(2)}(r,0) = -\frac{1}{\pi\mu(\lambda+2\mu)}\int_r^\infty [2\mu^2 A(t)+(\lambda+3\mu)D(t)]\frac{\mathrm{d}t}{\sqrt{t^2-r^2}}, \quad r>0. \tag{14}$$

Also, eqns (1)–(2) yield

$$r[u_r^{(1)}(r,0)-u_r^{(2)}(r,0)] = \frac{2}{\pi}\int_0^r \frac{\rho A(\rho)\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}, \quad r>0, \tag{15}$$

$$u_z^{(1)}(r,0)-u_z^{(2)}(r,0) = \frac{2}{\pi}\int_0^r \frac{B(\rho)\,\mathrm{d}\rho}{\sqrt{r^2-\rho^2}}, \quad r>0. \tag{16}$$

Thus, the stress and displacement components on the plane $z = 0$ are given in terms of the functions A, B, C, D by eqns (7), (8) and (12)–(16). As in Parihar and Rao (1991), a discussion on the stress components $\sigma_{rr}(r, z)$ and $\sigma_{\theta\theta}(r, z)$ is relegated to the last section.

3. THE EXTERNAL CIRCULAR CRACK

Let an external circular crack be located in the plane $z = 0$ of a homogeneous and isotropic elastic solid. In terms of the cylindrical coordinates (r, θ, z) the crack occupies the region $r \geqslant a$ $(z = 0)$. In other words, we consider the stress distribution in two half spaces connected by the circular region of radius a with the crack covering the region $z = 0^\pm$ and $r \geqslant a$. The crack is subjected to axisymmetric surface tractions which are not necessarily self-equilibrating. The continuity and the boundary conditions on the plane $z = 0$ may be written

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0), \quad u_z^{(1)}(r,0) = u_z^{(2)}(r,0), \quad 0 \leqslant r < a, \tag{17}$$

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0), \quad \sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0), \quad 0 \leqslant r < a, \tag{18}$$

$$\sigma_{zz}^{(1)}(r,0)+\sigma_{zz}^{(2)}(r,0) = H^*(r), \quad r>a, \tag{19}$$

$$\sigma_{zz}^{(1)}(r,0)-\sigma_{zz}^{(2)}(r,0) = Q^*(r), \quad r>a, \tag{20}$$

$$\sigma_{rz}^{(1)}(r,0)+\sigma_{rz}^{(2)}(r,0) = G^*(r), \quad r>a, \tag{21}$$

$$\sigma_{rz}^{(1)}(r,0)-\sigma_{rz}^{(2)}(r,0) = P^*(r), \quad r>a, \tag{22}$$

where H^*, G^*, F^*, Q^* are the known functions representing the surface tractions on the crack.

In view of the boundary conditions (19) and (21) eqns (7) and (8) lead to the integral equations

$$rH^*(r) = \frac{2\mu}{\pi(\lambda+2\mu)}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^\infty \frac{[2(\lambda+\mu)B(t)+C(t)]t\,\mathrm{d}t}{\sqrt{t^2-r^2}}, \quad r>a, \tag{23}$$

$$G^*(r) = \frac{2\mu}{\pi(\lambda+2\mu)}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^\infty \frac{[2(\lambda+\mu)A(t)-D(t)]\,\mathrm{d}t}{\sqrt{t^2-r^2}}, \quad r>a. \tag{24}$$

These are Abel integral equations of the type given by eqns (4.21)–(4.22) of Parihar and Rao (1991) and thus we have

$$2(\lambda+\mu)B(t)+C(t) = -\frac{(\lambda+2\mu)}{\mu}\int_t^\infty \frac{rH^*(r)\,\mathrm{d}r}{\sqrt{r^2-t^2}}, \quad t>a, \tag{25}$$

$$2(\lambda+\mu)A(t)-D(t) = -\frac{(\lambda+2\mu)}{\mu}\int_t^\infty \frac{tG^*(r)\,\mathrm{d}r}{\sqrt{r^2-t^2}}, \quad t>a. \tag{26}$$

Integrating eqn (5) with respect to ρ and making use of the conditions (6), (18) and (22) we find

$$C(\rho) = \begin{Bmatrix} \mathbb{P}(\rho), & \rho>a \\ -\int_a^\infty P^*(s)\,\mathrm{d}s, & 0<\rho<a \end{Bmatrix}, \tag{27}$$

where

$$\mathbb{P}(\rho) = \int_a^\rho \frac{\rho P^*(s)\,\mathrm{d}s}{\sqrt{\rho^2-s^2}} - \int_a^\infty P^*(s)\,\mathrm{d}s, \quad \rho>a. \tag{28}$$

Also, if we make use of the continuity conditions (17) and (18), eqns (1), (2) and (4) yield

$$A(\rho)=0, \quad B(\rho)=0, \quad D(\rho)=0, \quad 0<\rho<a. \tag{29}$$

Finally, eqn (4) together with (18) and (20) gives

$$D(\rho) = \int_a^\rho \frac{sQ^*(s)\,\mathrm{d}s}{\sqrt{\rho^2-s^2}}, \quad \rho>a. \tag{30}$$

Thus, the unknown functions A, B, C, D on the interval $(0, \infty)$ are given by eqns (25)–(30) in terms of the prescribed functions H^*, G^*, P^*, Q^*.

4. STRESS AND DISPLACEMENT COMPONENTS ON THE CRACK PLANE

In this section we calculate explicit expressions for stress and displacement components on the crack plane in terms of the prescribed functions H^*, G^*, P^*, Q^*. Using (25), (27), (29), (A2) and (A3) we can show that

$$\int_r^a \frac{[2(\lambda+\mu)B(t)+C(t)]t\,dt}{\sqrt{t^2-r^2}} = -\sqrt{a^2-r^2}\int_a^\infty P^*(s)\,ds, \quad 0<r<a, \tag{31}$$

$$\int_a^\infty \frac{[2(\lambda+\mu)B(t)+C(t)]t\,dt}{\sqrt{t^2-r^2}} = -\frac{(\lambda+2\mu)}{\mu}\int_a^\infty sH^*(s)\sin^{-1}\sqrt{\frac{s^2-a^2}{s^2-r^2}}\,ds, \quad 0<r<a. \tag{32}$$

In view of (31)–(32), eqn (7) yields

$$\sigma_{zz}^{(1)}(r,0)+\sigma_{zz}^{(2)}(r,0) = \frac{2\mu}{\pi(\lambda+2\mu)\sqrt{a^2-r^2}}\int_a^\infty P^*(s)\,ds - \frac{2}{\pi\sqrt{a^2-r^2}}\int_a^\infty \frac{s\sqrt{s^2-a^2}H^*(s)\,ds}{s^2-r^2}, \quad 0<r<a. \tag{33}$$

In an analogous manner we can show that eqn (8) gives

$$\sigma_{rz}^{(1)}(r,0)+\sigma_{rz}^{(2)}(r,0) = -\frac{2r}{\pi\sqrt{a^2-r^2}}\int_a^\infty \frac{\sqrt{s^2-a^2}G^*(s)\,ds}{s^2-r^2}, \quad 0<r<a. \tag{34}$$

Substituting $A(\rho)$ from (26), (29) and (30) into (15), interchanging the order of integrations in the term containing Q^* and using the integrals (A2)–(A3) we get

$$r[u_r^{(1)}(r,0)-u_r^{(2)}(r,0)] = -\frac{1}{\pi(\lambda+\mu)}\left[\frac{(\lambda+2\mu)}{\mu}\int_a^r \frac{\rho^2\,d\rho}{\sqrt{r^2-\rho^2}}\int_\rho^\infty \frac{G^*(s)\,ds}{\sqrt{s^2-\rho^2}} - \frac{\pi}{2}\int_a^r sQ^*(s)\,ds\right], \quad r>a. \tag{35}$$

Similarly, using (25), (27), (29), (A2) and (A3) we find

$$u_z^{(1)}(r,0)-u_z^{(2)}(r,0) = -\frac{(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}\int_a^r \frac{d\rho}{\sqrt{r^2-\rho^2}}\int_\rho^\infty \frac{sH^*(s)\,ds}{\sqrt{s^2-\rho^2}} + \frac{1}{2(\lambda+\mu)}\left[\int_r^\infty P^*(s)\,ds - \frac{2}{\pi}\sin^{-1}\left(\frac{a}{r}\right)\int_a^\infty P^*(s)\,ds\right], \quad r>a. \tag{36}$$

Therefore, the jumps in the displacement components on the crack plane are given by the relations (35), (36) and (17).

In order to complete the calculations of the displacement components on the crack plane we consider the expressions (12)–(14). Using (14), (29), (30) and (26) we can write

$$u_z^{(1)}(r,0)+u_z^{(2)}(r,0) = \frac{1}{\pi\mu(\lambda+\mu)}\left[\mu\int_{\max(r,a)}^\infty G^*(s)\beta(r,s)\,ds - (\lambda+2\mu)\int_{\max(r,a)}^\infty \frac{dt}{\sqrt{t^2-r^2}}\int_a^t \frac{sQ^*(s)\,ds}{\sqrt{t^2-s^2}}\right], \quad r>0, \tag{37}$$

where the function $\beta(r,s)$ is defined by (A2). Similarly, using (13), (25), (27) and (29) we can write

$$\Delta(r) = \frac{(\lambda+3\mu)}{\pi\mu(\lambda+2\mu)} H_0(a-r)\sqrt{a^2-r^2}\int_a^\infty P^*(s)\,\mathrm{d}s - \frac{1}{\pi(\lambda+\mu)}\int_{\max(r,a)}^\infty sH^*(s)\beta(r,s)\,\mathrm{d}s$$

$$-\frac{(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}\int_{\max(r,a)}^\infty \frac{t\mathbb{P}(t)\,\mathrm{d}t}{\sqrt{t^2-r^2}}, \quad r>0, \quad (38)$$

where $H_0(x)$ is the Heaviside unit step function (Sneddon 1979), and the integrals $\beta(r, s)$ and $\mathbb{P}(t)$ are defined by (A2) and (28). For $r = 0$, the last integral in (38) can be evaluated analytically [see eqns (A4) and (A5)] and thus we have

$$\Delta(0) = -\frac{1}{\pi(\lambda+\mu)}\left[\int_a^\infty sH^*(s)\cos^{-1}\left(\frac{a}{s}\right)\mathrm{d}s + \frac{a\mu}{(\lambda+2\mu)}\int_a^\infty P^*(s)\,\mathrm{d}s\right]. \quad (39)$$

Therefore, the radial components of displacement on the crack plane are given in terms of the prescribed functions H^*, G^*, P^*, Q^* by the relations (12), (38), (39), (35) and (17). The normal components of displacement are given by (37), (36) and (17). The normal and shear components of stress on the crack plane are given by (33), (34) and (18)–(22). Equations (33) and (34) together with (18) may be used to calculate the stress intensity factors at the rim of the crack.

The special case in which $Q^*(r) = G^*(r) = P^*(r) = 0$ for $r > a$ has been studied by Lowengrub and Sneddon (1965, 1969). The quantities given above by eqns (33) and (36) have also been calculated in Lowengrub and Sneddon (1965) and Sneddon and Lowengrub (1969), but they are given in a different form. With some further simplification of the expressions in Lowengrub and Sneddon (1965) and Sneddon and Lowengrub (1969) we can verify that the results are in complete agreement.

5. SPECIAL CASES OF THE LOADING FUNCTIONS

In the first two cases the crack surfaces are subjected to normal tractions alone, while in the remaining two cases the crack surfaces are subjected to shear tractions alone.

5.1. *Uniform stretching*

We first consider the case in which the prescribed functions in the boundary conditions (19)–(22) are given as

$$G^*(r) = 0, \quad P^*(r) = 0, \quad r > a, \quad (40)$$

$$H^*(r) = 0, \quad Q^*(r) = 0, \quad a < r < \varepsilon, \quad r > c, \quad (41)$$

$$H^*(r) = p_1, \quad Q^*(r) = p_2, \quad \varepsilon < r < c, \quad (42)$$

where p_1 and p_2 are constants. Physically, this corresponds to the case in which the solid is stretched in the positive z-direction on the upper crack surface and in the negative z-direction on the lower crack surface. The stretching is uniform and occurs over a circular ring of inner and outer radii ε and c, respectively.

Substituting the values (40)–(42) into the expressions (33), (34), (20) and (22) we get

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) = \frac{p_1}{\pi}\left[\tan^{-1}\sqrt{\frac{c^2-a^2}{a^2-r^2}} - \tan^{-1}\sqrt{\frac{\varepsilon^2-a^2}{a^2-r^2}}\right.$$

$$\left.-(\sqrt{c^2-a^2}-\sqrt{\varepsilon^2-a^2})(a^2-r^2)^{-1/2}\right], \quad 0 \leqslant r < a, \quad (43)$$

and the components of shear stress vanish identically on the crack plane. The mode I and mode II stress intensity factors at the rim $(a, 0)$ of the crack are defined by

$$K_{\mathrm{I}} = \lim_{r\to a^-} \sqrt{2(a-r)}\sigma_{zz}(r,0), \quad K_{\mathrm{II}} = \lim_{r\to a^-} \sqrt{2(a-r)}\sigma_{rz}(r,0), \tag{44}$$

where the superscripts on the stress components have been omitted in view of their equality (18) for $0 \leqslant r < a$. Then using (40)–(43) we get

$$K_{\mathrm{I}} = -\frac{p_1}{\pi\sqrt{a}}[\sqrt{c^2-a^2}-\sqrt{\varepsilon^2-a^2}], \quad K_{\mathrm{II}} = 0. \tag{45}$$

Unlike the stress components, the normal components of displacement $u_z^{(1)}(r,0)$ and $u_z^{(2)}(r,0)$ cannot be written in terms of elementary functions. However, we can write them in terms of elliptic integrals defined by (B1) and (B2) in Appendix B. Substituting G^* and Q^* from (40)–(42) into (37) we get

$$u_z^{(1)}(r,0)+u_z^{(2)}(r,0) = -\frac{(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}p_2U_0(r), \quad 0 \leqslant r < \infty, \tag{46}$$

where the function U_0 is given by

$$U_0(r) = \begin{cases} U_1(r), & 0 \leqslant r < c \\ U_2(r), & r > c \end{cases} \tag{47}$$

together with

$$U_1(r) = \int_{\max(\varepsilon,r)}^{c} \frac{\sqrt{t^2-\varepsilon^2}\,\mathrm{d}t}{\sqrt{t^2-r^2}} + \int_c^\infty \frac{(\sqrt{t^2-\varepsilon^2}-\sqrt{t^2-c^2})\,\mathrm{d}t}{\sqrt{t^2-r^2}}, \quad 0 \leqslant r < c, \tag{48}$$

$$U_2(r) = \int_r^\infty \frac{(\sqrt{t^2-\varepsilon^2}-\sqrt{t^2-c^2})\,\mathrm{d}t}{\sqrt{t^2-r^2}}, \quad r > c. \tag{49}$$

Using the appropriate formulae (B3)–(B8) we find

$$U_1(r) = \begin{cases} cE\left(\frac{r}{c}\right)-\varepsilon E\left(\frac{r}{\varepsilon}\right), & 0 \leqslant r < \varepsilon \\ cE\left(\frac{r}{c}\right)-\frac{(\varepsilon^2-r^2)}{r}K\left(\frac{\varepsilon}{r}\right)-rE\left(\frac{\varepsilon}{r}\right), & \varepsilon < r < c \end{cases}, \tag{50}$$

$$U_2(r) = \frac{(r^2-\varepsilon^2)}{r}K\left(\frac{\varepsilon}{r}\right)-rE\left(\frac{\varepsilon}{r}\right)-\frac{(r^2-c^2)}{r}K\left(\frac{c}{r}\right)+rE\left(\frac{c}{r}\right), \quad r > c. \tag{51}$$

Thus, the sum of the normal components of displacement on the crack plane is given in terms of elliptic integrals by the relations (46), (47), (50) and (51). In order to calculate the difference, we substitute H^* and P^* from (40)–(42) into (36) to get

$$u_z^{(1)}(r,0)-u_z^{(2)}(r,0) = -\frac{(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}p_1V_0(r), \quad r > a, \tag{52}$$

where the function V_0 for two subintervals is given by

$$V_0(r) = \int_a^r \frac{(\sqrt{c^2-t^2}-\sqrt{\varepsilon^2-t^2})\,\mathrm{d}t}{\sqrt{r^2-t^2}}, \quad a \leqslant r < \varepsilon, \tag{53}$$

$$V_0(r) = \int_a^{\min(r,c)} \frac{\sqrt{c^2-t^2}\,\mathrm{d}t}{\sqrt{r^2-t^2}} - \int_a^{\varepsilon} \frac{\sqrt{\varepsilon^2-t^2}\,\mathrm{d}t}{\sqrt{r^2-t^2}}, \quad \varepsilon \leqslant r < \infty. \tag{54}$$

By using the substitution $t = r\sin\theta$, the integral in (53) may be written in terms of elliptic integrals as follows:

$$V_0(r) = c\left[E\left(\frac{r}{c}\right)-E\left(\sin^{-1}\frac{a}{r},\frac{r}{c}\right)\right]-\varepsilon\left[E\left(\frac{r}{\varepsilon}\right)-E\left(\sin^{-1}\frac{a}{r},\frac{r}{\varepsilon}\right)\right], \quad a \leqslant r < \varepsilon. \tag{55}$$

Similarly, the integrals in (54) may be written for two subintervals as

$$V_0(r) = c\left[E\left(\frac{r}{c}\right)-E\left(\sin^{-1}\frac{a}{r},\frac{r}{c}\right)\right]+\frac{(r^2-\varepsilon^2)}{r}\left[K\left(\frac{\varepsilon}{r}\right)-F\left(\sin^{-1}\frac{a}{\varepsilon},\frac{\varepsilon}{r}\right)\right]$$

$$-r\left[E\left(\frac{\varepsilon}{r}\right)-E\left(\sin^{-1}\frac{a}{\varepsilon},\frac{\varepsilon}{r}\right)\right], \quad \varepsilon < r < c, \tag{56}$$

$$V_0(r) = r\left[E\left(\frac{c}{r}\right)-E\left(\sin^{-1}\frac{a}{c},\frac{c}{r}\right)\right]-\frac{(r^2-c^2)}{r}\left[K\left(\frac{c}{r}\right)-F\left(\sin^{-1}\frac{a}{c},\frac{c}{r}\right)\right]$$

$$+\frac{(r^2-\varepsilon^2)}{r}\left[K\left(\frac{\varepsilon}{r}\right)-\left(\sin^{-1}\frac{a}{\varepsilon},\frac{\varepsilon}{r}\right)\right]-r\left[E\left(\frac{\varepsilon}{r}\right)-E\left(\sin^{-1}\frac{a}{\varepsilon},\frac{\varepsilon}{r}\right)\right], \quad r > c. \tag{57}$$

Therefore, the normal components of displacement $u_z^{(1)}(r,0)$ and $u_z^{(2)}(r,0)$ are given, in terms of elliptic integrals, by the relations (46), (47), (50)–(52) and (55)–(57). We can easily calculate the radial components of displacement $u_r^{(1)}(r,0)$ and $u_r^{(2)}(r,0)$ by using the relations (12), (38), (39), (28), (35) and (17) together with (40)–(42). The resulting expressions do not involve elliptic integrals.

The general case discussed above contains an interesting special case, namely, $\varepsilon = a$. We can deduce the results for this special case by setting $\varepsilon = a$ in the corresponding formulae given in this subsection for the general case.

5.2. *Concentrated normal ring load*

Now, we consider the case in which a normal load is distributed uniformly over the circumference of a circle of radius $c(c > a)$ on the upper surface of the crack. In addition a normal load, of magnitude different from that acting on the upper surface, is distributed uniformly over the circumference of a circle of radius c on the lower surface of the crack. We can deduce the results for this case from those in Subsection 5.1 by setting

$$\pi p_1(c^2-\varepsilon^2) = P_1, \quad \pi p_2(c^2-\varepsilon^2) = P_2 \tag{58}$$

and taking the limit as $\varepsilon \to c$. The parameters P_1 and P_2 represent the total loads and as such we keep them fixed while taking the limit as $\varepsilon \to c$. Thus, from (43) we get

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) = -\frac{P_1\sqrt{c^2-a^2}}{2\pi^2(c^2-r^2)\sqrt{a^2-r^2}}, \quad 0 \leqslant r < a. \tag{59}$$

The stress intensity factors corresponding to (45) are given by

$$K_{\rm I} = -P_1[2\pi^2\sqrt{a(c^2-a^2)}]^{-1}, \quad K_{\rm II} = 0. \tag{60}$$

The expressions for the normal components of displacement in Subsection 5.1 are given in terms of elliptic integrals. It is simpler to calculate the limit as $\varepsilon \to c$ from the integrals for the normal components of displacement given in the first instance. For example, taking the limit as $\varepsilon \to c$ in the expressions given by (46)–(49) together with (58) and converting the resulting integrals into elliptic integrals we get

$$u_z^{(1)}(r,0)+u_z^{(2)}(r,0) = -\frac{(\lambda+2\mu)}{2\pi^2\mu(\lambda+\mu)}P_2U_3(r), \quad r>0, \tag{61}$$

where

$$U_3(r) = \begin{Bmatrix} \dfrac{1}{c}K\left(\dfrac{r}{c}\right), & 0<r<c \\ \dfrac{1}{r}K\left(\dfrac{c}{r}\right), & r>c \end{Bmatrix}. \tag{62}$$

Similarly, taking the limit as $\varepsilon \to c$ in the expressions given by (52)–(54) together with (58) and converting the resulting integrals into elliptic integrals we find

$$u_z^{(1)}(r,0)-u_z^{(2)}(r,0) = -\frac{(\lambda+2\mu)}{2\pi^2\mu(\lambda+\mu)}P_1V_1(r), \quad r>a, \tag{63}$$

where

$$V_1(r) = \begin{Bmatrix} \dfrac{1}{c}\left[K\left(\dfrac{r}{c}\right)-F\left(\sin^{-1}\dfrac{a}{r},\dfrac{r}{c}\right)\right], & a<r<c \\ \dfrac{1}{r}\left[K\left(\dfrac{c}{r}\right)-F\left(\sin^{-1}\dfrac{a}{c},\dfrac{c}{r}\right)\right], & r>c \end{Bmatrix}. \tag{64}$$

Equations (61)–(64) together with (17) give the normal components of displacement for the case of concentrated normal ring load.

5.3. *Variable shear load*

Let us consider the case in which the prescribed functions H^*, G^*, P^*, Q^* in the boundary conditions (19)–(22) are given by

$$H^*(r) = 0, \quad Q^*(r) = 0, \quad r>a, \tag{65}$$

$$G^*(r) = 0, \quad P^*(r) = 0, \quad a<r<\varepsilon, \quad r>c, \tag{66}$$

$$G^*(r) = q_1\left(\frac{a}{r}\right)^2, \quad P^*(r) = q_2\left(\frac{a}{r}\right)^2, \quad \varepsilon<r<c, \tag{67}$$

where q_1 and q_2 are constants. Physically this corresponds to the case in which the solid is sheared radially away from the origin on the upper crack surface and toward the origin on the lower crack surface. The shearing action is variable over a circular ring of inner and outer radii ε and c respectively.

Substituting the values (65)–(67) into the expressions (33) and (34) for normal and shear stress components on the crack plane and making use of the relations (18) we get

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) = \frac{q_2\mu(c-\varepsilon)a^2}{\pi(\lambda+2\mu)c\varepsilon\sqrt{a^2-r^2}}, \quad 0<r<a, \tag{68}$$

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0) = -\frac{q_1a^2}{\pi r^2}\left[\frac{r}{\sqrt{a^2-r^2}}\left(\frac{\sqrt{c^2-a^2}}{c}-\frac{\sqrt{\varepsilon^2-a^2}}{\varepsilon}\right)\right.$$
$$\left.+\tan^{-1}\left(\frac{r\sqrt{\varepsilon^2-a^2}}{\varepsilon\sqrt{a^2-r^2}}\right)-\tan^{-1}\left(\frac{r\sqrt{c^2-a^2}}{c\sqrt{a^2-r^2}}\right)\right], \quad 0<r<a. \tag{69}$$

Using the formulae (44), we can write the mode I and mode II stress intensity factors for this case as

$$K_{\mathrm{I}} = \frac{q_2\mu(c-\varepsilon)a\sqrt{a}}{\pi(\lambda+2\mu)c\varepsilon}, \quad K_{\mathrm{II}} = -\frac{q_1\sqrt{a}}{\pi}\left[\frac{\sqrt{c^2-a^2}}{c}-\frac{\sqrt{\varepsilon^2-a^2}}{\varepsilon}\right]. \tag{70}$$

Next, substituting the values (65)–(67) into the expression (36) yields

$$u_z^{(1)}(r,0)-u_z^{(2)}(r,0) = \frac{q_2a^2}{\pi(\lambda+\mu)}\left[V_2(r)-\frac{(c-\varepsilon)}{c\varepsilon}\sin^{-1}\left(\frac{a}{r}\right)\right], \quad r>a, \tag{71}$$

where, in terms of the signum function, we have

$$V_2(r) = \frac{\pi}{4c}\left[\frac{(c-\varepsilon)}{\varepsilon}+\frac{(c-r)}{r}\operatorname{sgn}(c-r)-\frac{c(\varepsilon-r)}{r\varepsilon}\operatorname{sgn}(\varepsilon-r)\right], \quad r>a. \tag{72}$$

Also, substituting the values (65)–(67) into (37) and making use of the relation (A3) we find

$$u_z^{(1)}(r,0)+u_z^{(2)}(r,0) = \left\{\begin{matrix} U_4(r), & 0<r<a \\ q_1a^2V_2(r)[\pi(\lambda+\mu)]^{-1}, & r>a \end{matrix}\right\}, \tag{73}$$

where $V_2(r)$ is given by (72) and we have

$$U_4(r) = \frac{q_1a^2}{\pi(\lambda+\mu)}\left[\frac{1}{\varepsilon}\sin^{-1}\sqrt{\frac{\varepsilon^2-a^2}{\varepsilon^2-r^2}}-\frac{1}{c}\sin^{-1}\sqrt{\frac{c^2-a^2}{c^2-r^2}}\right.$$
$$\left.-\frac{1}{r}\tan^{-1}\left(\frac{r\sqrt{\varepsilon^2-a^2}}{\varepsilon\sqrt{a^2-r^2}}\right)+\frac{1}{r}\tan^{-1}\left(\frac{r\sqrt{c^2-a^2}}{c\sqrt{a^2-r^2}}\right)\right], \quad 0<r<a. \tag{74}$$

Therefore, the normal components of displacement $u_z^{(1)}(r,0)$ and $u_z^{(2)}(r,0)$ on the crack plane are given by the relations (71)–(74) together with (17). Unlike the stress components and the normal components of displacement, the radial components of displacement $u_r^{(1)}(r,0)$ and $u_r^{(2)}(r,0)$ cannot be written in terms of elementary functions. However, we can write them in terms of elliptic integrals of the first and the second kind defined by (B1) and (B2). The analysis is analogous to that for the normal components of displacement in Subsection 5.1.

The problem discussed in this subsection contains three other problems as special cases, namely, (i) $\varepsilon = a$ and c finite; (ii) $\varepsilon > a$ and $c = \infty$; (iii) $\varepsilon = a$ and $c = \infty$. Case (iii) is a case of radially decaying shear load on the crack surfaces. We can deduce the results for these three cases from the results obtained above in this subsection by setting $\varepsilon = a$ and/or letting $c \to \infty$.

5.4. *Concentrated shear ring load*

Let us consider the case in which a shear load is distributed uniformly over the circumference of a circle of radius $c(c > a)$ on the upper surface of the crack. In addition, a shear load of magnitude different from that acting on the upper surface is distributed uniformly over the circumference of a circle of radius c on the lower surface of the crack. We can deduce the results for this case from those in Subsection 5.3 by setting

$$2\pi q_1 a^2 \log(c/\varepsilon) = Q_1, \quad 2\pi q_2 a^2 \log(c/\varepsilon) = Q_2 \tag{75}$$

and taking the limit as $\varepsilon \to c$. The parameters Q_1 and Q_2 represent total loads calculated from the formulae

$$Q_1 = \int_0^{2\pi} \int_0^{\infty} G^*(r) r \, \mathrm{d}r \, \mathrm{d}\theta, \quad Q_2 = \int_0^{2\pi} \int_0^{\infty} P^*(r) r \, \mathrm{d}r \, \mathrm{d}\theta \tag{76}$$

together with (66)–(67). We keep Q_1 and Q_2 fixed while taking the limit as $\varepsilon \to c$. The limiting values of shear and normal stress components given by (69) and (68) may be written as

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0) = -\frac{Q_1 r\sqrt{c^2-a^2}}{2\pi^2 c(c^2-r^2)\sqrt{a^2-r^2}}, \quad 0 < r < a, \tag{77}$$

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) = \frac{\mu Q_2}{2\pi^2(\lambda+2\mu)c\sqrt{a^2-r^2}}, \quad 0 < r < a. \tag{78}$$

We can find the corresponding stress intensity factors by using the formulae (44). The mode I and mode II stress intensity factors K_{I} and K_{II} are given by

$$K_{\mathrm{I}} = \frac{\mu Q_2 \sqrt{a}}{2\pi^2(\lambda+2\mu)ac}, \quad K_{\mathrm{II}} = -\frac{Q_1\sqrt{a}}{2\pi^2 c\sqrt{c^2-a^2}}. \tag{79}$$

Taking the limit as $\varepsilon \to c$ in the expression given by the relations (71), (72) and (75) we get

$$u_z^{(1)}(r,0) - u_z^{(2)}(r,0) = \frac{Q_2}{8\pi(\lambda+\mu)c}\left[1+\operatorname{sgn}(c-r) - \frac{4}{\pi}\sin^{-1}\left(\frac{a}{r}\right)\right], \quad r > a. \tag{80}$$

Similarly, the limiting values as $\varepsilon \to c$ of the expression given by the relations (73), (74), (72) and (75) may be written as

$$u_z^{(1)}(r,0) + u_z^{(2)}(r,0) = \frac{Q_1}{8\pi(\lambda+\mu)c} U_5(r), \quad r > 0, \tag{81}$$

where the function U_5 is given by

$$U_5(r) = \left\{ \begin{array}{ll} \frac{4}{\pi}\sin^{-1}\sqrt{(c^2-a^2)/(c^2-r^2)}, & 0 < r < a \\ 1+\operatorname{sgn}(c-r), & r > a \end{array} \right\}. \tag{82}$$

Thus, the normal components of displacement $u_z^{(1)}(r, 0)$ and $u_z^{(2)}(r, 0)$ on the crack plane for the concentrated shear ring load are given by the relations (80)–(82) and (17).

5.5. *Numerical results*

The stress intensity factors and the stress components on the crack plane derived in Subsections 5.1–5.4 are in terms of elementary functions and need no further elaboration. However, the normal components of displacement in Subsections 5.1–5.2 are in terms of complete and incomplete elliptic integrals of the first and the second kind. The displacement components are further complicated by the fact that they have different expressions for different subintervals, namely, $(0, a)$, (a, ε), (ε, c) and (c, ∞). At the end points of these subintervals the expressions are generally not valid; but the limit as we approach to any of the end points, through the interior points of the appropriate interval, can be easily carried out and thus the continuity of displacement components at the end points can be established analytically. Of course, the displacement components should not be expected to be continuous at points where concentrated forces are acting (Subsections 5.2 and 5.4). Indeed the displacement components may be unbounded at these points (Timoshenko and Goodier, 1970).

In order to illustrate the use of the formulae derived in Subsections 5.1–5.4 we have carried out the numerical computations for the normal components of displacement. The attention is restricted to the quantity of practical interest, namely, the crack opening displacements. For the numerical calculations we consider the following particular case of loading:

$$\sigma_{zz}^{(1)}(r,0) = -p_0, \quad \varepsilon < r < c, \tag{83}$$

$$\sigma_{zz}^{(1)}(r,0) = 0, \quad a < r < \varepsilon, \quad r > c, \tag{84}$$

$$\sigma_{rz}^{(1)}(r,0) = 0, \quad \sigma_{rz}^{(2)}(r,0) = 0, \quad \sigma_{zz}^{(2)}(r,0) = 0, \quad r > a, \tag{85}$$

where p_0 is a positive constant. This is a special case of the loading studied in Subsection 5.1 with

$$p_1 = -p_0, \quad p_2 = -p_0. \tag{86}$$

In this case the lower surface of the crack is free from tractions while the upper surface is subjected only to a uniform normal traction over a circular ring of inner and outer radii ε and c. The sum of crack opening displacements $u_z^{(1)}(r, 0)$ and $u_z^{(2)}(r, 0)$ is given by the relations (46), (47), (50), (51) and (86). We can find the sum at an end point by taking the limit as $r \to \varepsilon$ or as $r \to c$. For example, at $r = c$ we have

$$u_z^{(1)}(c,0)+u_z^{(2)}(c,0) = \frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\frac{(c^2-\varepsilon^2)}{c}K\left(\frac{\varepsilon}{c}\right)-cE\left(\frac{\varepsilon}{c}\right)+c\right]. \tag{87}$$

The difference between the crack opening displacements $u_z^{(1)}(r, 0)$ and $u_z^{(2)}(r, 0)$ is given by the relations (52), (55)–(57) and (86). Again, we can find the difference at an end point by taking the limit as $r \to \varepsilon$ and as $r \to c$. For example, at $r = c$ we have

$$u_z^{(1)}(c,0)-u_z^{(2)}(c,0) = \frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left\{c-a+\frac{(c^2-\varepsilon^2)}{c}\left[K\left(\frac{\varepsilon}{c}\right)-F\left(\sin^{-1}\frac{a}{\varepsilon},\frac{\varepsilon}{c}\right)\right]\right.$$
$$\left.-c\left[E\left(\frac{\varepsilon}{c}\right)-E\left(\sin^{-1}\frac{a}{\varepsilon},\frac{\varepsilon}{c}\right)\right]\right\}. \tag{88}$$

Let us introduce a parameter α by the definition

$$\alpha = 2\pi\mu(\lambda+\mu)[(\lambda+2\mu)ap_0]^{-1} = \pi\hat{E}[2(1-v^2)ap_0]^{-1}, \tag{89}$$

where v denotes Poisson's ratio and $\hat{E}$ the Young's modulus of the material. We note that the dimensionless quantities $\alpha u_z^{(1)}(r,0)$ and $\alpha u_z^{(2)}(r,0)$ are independent of the material constants. We can calculate the elliptic integrals by using the algorithms given in Spanier and Oldham (1987). The algorithms are based on Landen's descending transformation [see also Abramowitz and Stegun (1965)]. The variation of $\alpha u_z^{(1)}(r,0)$ and $\alpha u_z^{(2)}(r,0)$ with r/a is shown in Fig. 2 for $c/a = 1.2$ and $\varepsilon/a = 1.00, 1.08, 1.18$. The normal component of displacement $\alpha u_z^{(1)}(r,0)$ for the upper crack surface rises in the beginning, attains a maximum and then decreases with r/a. The curve for $\alpha u_z^{(2)}(r,0)$ the normal displacement component for the lower crack surface remains close to the crack plane. The variation for $\alpha u_z^{(1)}(r,0)$ and $\alpha u_z^{(2)}(r,0)$ with r/a is shown in Fig. 3 for $c/a = 1.6$ and $\varepsilon/a = 1.00, 1.20, 1.56$. Since $1 \leqslant \varepsilon/a < c/a$, larger values of ε/a could be taken in this figure. The variation of $\alpha u_z^{(1)}(r,0)$ and $\alpha u_z^{(2)}(r,0)$ with r/a is shown in Fig. 4 for $c/a = 2.0$ and $\varepsilon/a = 1.0, 1.4, 1.9$. The total applied load P_0 which is given by

$$P_0 = \pi(c^2-\varepsilon^2)p_0, \tag{90}$$

increases with c for a fixed value of ε. Therefore we cannot increase c indefinitely.

Next, we consider the special case of concentrated normal ring load discussed in Subsection 5.2. For numerical calculations we have taken

$$P_1 = P_2 = -P_0. \tag{91}$$

Physically, this corresponds to the case in which a normal load of magnitude P_0 is distributed uniformly over the circumference of a circle of radius c on the upper crack surface while the lower crack surface is free from tractions. For this case, the normal components

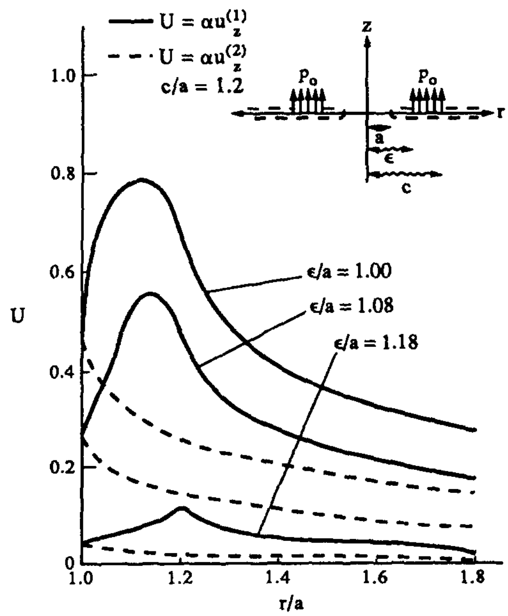

Fig. 2. Variation with r/a ($1 \leqslant r/a < \infty$) of the normal components of displacement $\alpha u_z^{(1)}(r,0)$ and $\alpha u_z^{(2)}(r,0)$ on the crack plane when $c/a = 1.2$. The constant α is defined by eqn (89) and the problem geometry is included in this figure. Curves are drawn for $\varepsilon/a = 1.00, 1.08$ and 1.18.

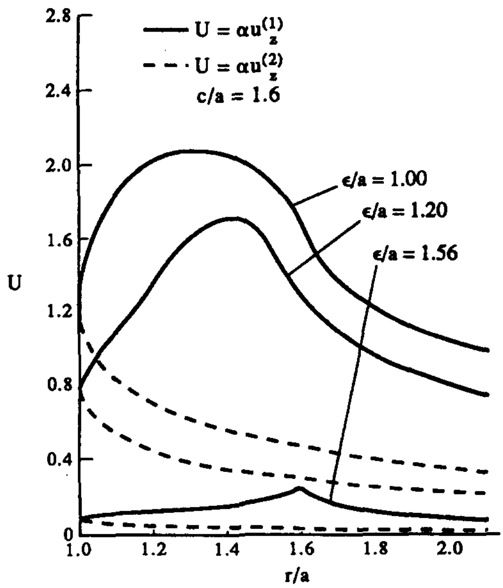

Fig. 3. Variation with r/a ($1 \leqslant r/a < \infty$) of the normal components of displacement $\alpha u_z^{(1)}(r, 0)$ and $\alpha u_z^{(2)}(r, 0)$ on the crack plane when $c/a = 1.6$. The constant α and the problem geometry are the same as in Fig. 1. Curves are drawn for $\varepsilon/a = 1.00$, 1.20 and 1.56.

$u_z^{(1)}(r, 0)$ and $u_z^{(2)}(r, 0)$ of displacement are given by the relations (61)–(64) and (91). We note that $u_z^{(1)}(r, 0) \to \infty$ as $r \to c$, but $u_z^{(2)}(c, 0)$ is finite and is given by

$$u_z^{(2)}(c, 0) = \frac{(\lambda+2\mu)P_0}{8\pi^2\mu(\lambda+\mu)c}\log\left(\frac{c+a}{c-a}\right). \tag{92}$$

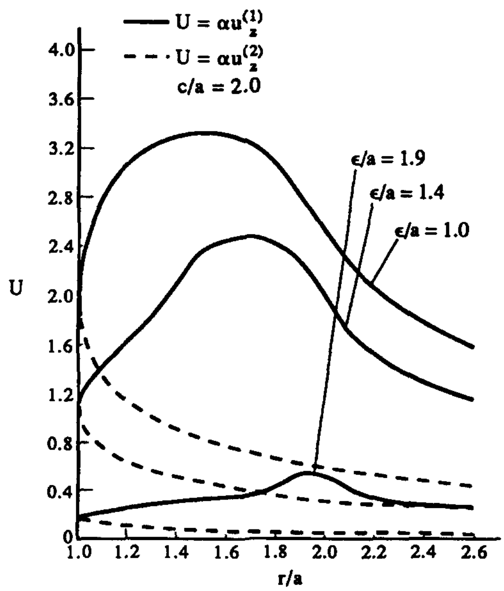

Fig. 4. Variation with r/a ($1 \leqslant r/a < \infty$) of the normal components of displacement $\alpha u_z^{(1)}(r, 0)$ and $\alpha u_z^{(2)}(r, 0)$ on the crack plane when $c/a = 2.0$. The constant α and the problem geometry are the same as in Fig. 1. Curves are drawn for $\varepsilon/a = 1.0$, 1.4 and 1.9.

We introduce a parameter β by the definition

$$\beta = 2\pi^2\mu(\lambda+\mu)a[(\lambda+2\mu)P_0]^{-1} = \pi^2\hat{E}a[2(1-\nu^2)P_0]^{-1}, \tag{93}$$

and note that the dimensionless quantities $\beta u_z^{(1)}(r, 0)$ and $\beta u_z^{(2)}(r, 0)$ are independent of the material constants. The variation of $\beta u_z^{(1)}(r, 0)$ and $\beta u_z^{(2)}(r, 0)$ with r/a is shown in Fig. 5 for $c/a = 1.2, 1.6, 2.0$.

Finally, we take up the case in which an inclined load of magnitude P_0 is distributed uniformly over a circle of radius c on the upper surface of the crack while the lower surface is free from tractions. If the inclination of the load with the radial direction is θ_0, we can find the results for this case by applying a concentrated normal ring load of magnitude $P_0 \sin\theta_0$ and a concentrated shear ring load of magnitude $P_0 \cos\theta_0$, that is, by setting

$$P_1 = P_2 = -P_0 \sin\theta_0, \quad Q_1 = Q_2 = -P_0 \cos\theta_0, \tag{94}$$

and superimposing the results obtained in Subsections 5.2 and 5.4. In this sense the results shown in Fig. 5 are those for $\theta_0 = \pi/2$. The normal components of displacement $u_z^{(1)}(r, 0)$ and $u_z^{(2)}(r, 0)$ for the case of concentrated shear ring load are given by the relations (80)–(82) and (94). We have superimposed these expressions on those given by the relations (61)–(64) and (94). The variation of resulting dimensionless quantities $\beta u_z^{(1)}(r, 0)$ and $\beta u_z^{(2)}(r, 0)$ with r/a is shown in Fig. 6 for $c/a = 1.2, 1.6, 2.0$ and $\nu = 0$, where β is defined by (93). We note that the formulae for concentrated inclined ring load are not independent of the material constants as they depend on Poisson's ratio ν. However, the results are dominated by the normal component of the concentrated inclined ring load and a change in the values of ν brings about a marginal change in the results.

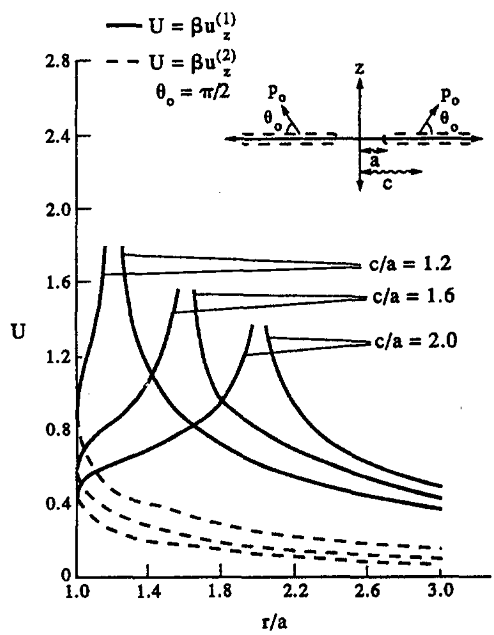

Fig. 5. Variation with r/a ($1 \leqslant r/a < c/a$ and $c/a < r/a < \infty$) of the normal components of displacement $\beta u_z^{(1)}(r, 0)$ and $\beta u_z^{(2)}(r, 0)$ on the crack plane for the normal concentrated ring load problem. The constant β is defined by eqn (93) and the problem geometry is included in this figure. Curves are drawn for $c/a = 1.2, 1.6$ and 2.0.

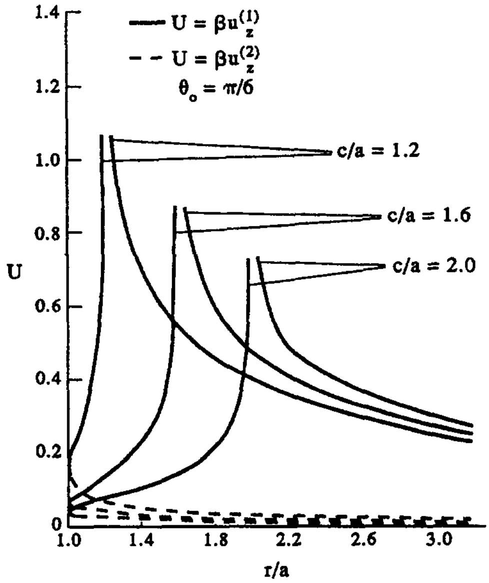

Fig. 6. Variation with r/a ($1 \leqslant r/a < c/a$ and $c/a < r/a < \infty$) of the normal components of displacement $\beta u_z^{(1)}(r, 0)$ and $\beta u_z^{(2)}(r, 0)$ on the crack plane for the inclined concentrated ring load problem. The constant β, the values assigned to c/a and the problem geometry (with $\theta_0 = \pi/6$) are the same as in Fig. 4. However, unlike Figs 2–5, the results depend on Poisson's ratio ν which is taken to be zero.

6. THE REMAINING STRESS COMPONENTS

It was indicated, at the end of Section 2, that a discussion on the stress components $\sigma_{rr}(r, z)$ and $\sigma_{\theta\theta}(r, z)$ would be taken up in a later section. As in the earlier sections we restrict our attention to the limiting values as $z \to 0^+$ and as $z \to 0^-$. In terms of the functions, A, B, C, D, these limiting values are given in Parihar and Rao (1991) [see eqns (8.7)–(8.14)]. Our aim, in this section is to show that these limiting values may be written in terms of the radial components of displacement and the normal components of stress on the crack plane. Set

$$S(r) = u_r^{(1)}(r, 0) - u_r^{(2)}(r, 0), \quad r > 0, \tag{95}$$

$$V(r) = u_r^{(1)}(r, 0) + u_r^{(2)}(r, 0), \quad r > 0, \tag{96}$$

then eqn (1) may be written as

$$\frac{\partial}{\partial \rho} \int_0^\rho \frac{\rho S(r)\, \mathrm{d}r}{\sqrt{\rho^2 - r^2}} = A(\rho), \quad \rho > 0. \tag{97}$$

Carrying out an integration by parts yields

$$\int_0^\rho \frac{\mathrm{d}}{\mathrm{d}r}[rS(r)] \frac{\mathrm{d}r}{\sqrt{\rho^2 - r^2}} = A(\rho), \quad \rho > 0. \tag{98}$$

Using (97) and (98) we can write eqn (8.13) of Parihar and Rao (1991) in the form

$$R_3(\rho) = \mu \int_0^\rho \left\{ \frac{d}{dr}[rS(r)] - \frac{2}{r}[rS(r)] \right\} \frac{dr}{\sqrt{\rho^2 - r^2}}, \quad \rho > 0. \tag{99}$$

In a similar manner, we can write R_4 given by eqn (8.14) of Parihar and Rao (1991) in terms of V. In view of (96) we can write eqn (3.37) of Parihar and Rao (1991) in the form

$$\frac{\partial}{\partial \rho} \int_0^\rho \frac{\rho V(r)\, dr}{\sqrt{\rho^2 - r^2}} = -\frac{1}{2\pi\mu(\lambda + 2\mu)} \int_{-\infty}^{\infty} \frac{[2\mu^2 B(t) - (\lambda + 3\mu) C(t)]\, dt}{\rho - t}, \quad \rho > 0. \tag{100}$$

If we follow the steps used in arriving at eqn (99) from (97), we can show that

$$R_4(\rho) = \mu \int_0^\rho \left\{ \frac{d}{dr}[rV(r)] - \frac{2}{r}[rV(r)] \right\} \frac{dr}{\sqrt{\rho^2 - r^2}}, \quad \rho > 0. \tag{101}$$

Now, using the relations (99) and (101) given above and eqns (8.9)–(8.10) of Parihar and Rao (1991) we get

$$r[\sigma_{rr}^{(1)}(r,0) - \sigma_{\theta\theta}^{(1)}(r,0)] = \mu \left\{ \frac{d}{dr}[rV(r) + rS(r)] - \frac{2}{r}[rV(r) + rS(r)] \right\}, \quad r > 0, \tag{102}$$

$$r[\sigma_{rr}^{(2)}(r,0) - \sigma_{\theta\theta}^{(2)}(r,0)] = \mu \left\{ \frac{d}{dr}[rV(r) - rS(r)] - \frac{2}{r}[rV(r) - rS(r)] \right\}, \quad r > 0. \tag{103}$$

Also, using (98) and (4) we can write eqn (8.11) of Parihar and Rao (1991) in the form

$$R_1(\rho) = \int_0^\rho \left\{ \frac{\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} \frac{d}{dr}[rS(r)] + \frac{\lambda r}{(\lambda + 2\mu)} [\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0)] \right\} \frac{dr}{\sqrt{\rho^2 - r^2}}, \quad \rho > 0. \tag{104}$$

Similarly, using (100) given above and eqns (3.35) and (8.12) of Parihar and Rao (1991) we can show that

$$R_2(\rho) = \int_0^\rho \left\{ \frac{\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} \frac{d}{dr}[rV(r)] + \frac{\lambda r}{(\lambda + 2\mu)} [\sigma_{zz}^{(1)}(r,0) + \sigma_{zz}^{(2)}(r,0)] \right\} \frac{dr}{\sqrt{\rho^2 - r^2}}, \quad \rho > 0. \tag{105}$$

In view of the above two relations and those given by eqns (8.7)–(8.8) of Parihar and Rao (1991) we obtain

$$r[\sigma_{rr}^{(1)}(r,0) + \sigma_{\theta\theta}^{(1)}(r,0)] = \frac{\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} \frac{d}{dr}[rV(r) + rS(s)] + \frac{2\lambda r}{(\lambda + 2\mu)} \sigma_{zz}^{(1)}(r,0), \quad r > 0, \tag{106}$$

$$r[\sigma_{rr}^{(2)}(r,0) + \sigma_{\theta\theta}^{(2)}(r,0)] = \frac{\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} \frac{d}{dr}[rV(r) - rS(r)] + \frac{2\lambda r}{(\lambda + 2\mu)} \sigma_{zz}^{(2)}(r,0), \quad r > 0. \tag{107}$$

Therefore, in any problem, once $u_r^{(1)}(r, 0)$, $u_r^{(2)}(r, 0)$, $\sigma_{zz}^{(1)}(r, 0)$ and $\sigma_{zz}^{(2)}(r, 0)$ are determined, the stress components $\sigma_{rr}^{(1)}(r, 0)$, $\sigma_{rr}^{(2)}(r, 0)$, $\sigma_{\theta\theta}^{(1)}(r, 0)$ and $\sigma_{\theta\theta}^{(2)}(r, 0)$ are given by the relations (102), (103), (106), (107), (95) and (96). For example, we can easily verify this statement for the penny shaped crack problem discussed in Parihar and Rao (1991). In the case of external crack problem discussed in earlier sections, the quantities $u_r^{(1)}(r, 0)$, $u_r^{(2)}(r, 0)$, $\sigma_{zz}^{(1)}(r, 0)$ and $\sigma_{zz}^{(2)}(r, 0)$ are given by the relations (35), (12), (38), (A3), (28), (39), (33) and (17)–(20) in terms of the prescribed functions H^*, G^*, P^*, Q^*.

7. CONCLUDING REMARKS

(a) The expressions for Abel transforms of stress and displacement components valid for the upper half space ($z > 0$) and the lower half space ($z < 0$) together with their limiting values as $z \to 0^+$ and as $z \to 0^-$ are given in Parihar and Rao (1991) [see eqns (3.29)–(3.38)] in terms of the functions A, B, C, D defined by (1)–(4). The functions A, B, C, D are Abel transforms of the displacement and stress discontinuities on the plane $z = 0$. These expressions have been used in Parihar and Rao (1991) to solve a penny shaped crack problem, and the same expressions have been used in the present paper to solve an external crack problem. We find that the expressions are more suitable for the external crack problem than for the penny shaped crack problem.

(b) From the relations (33), (34), (18)–(22) and (44) we note that when the crack surfaces are subjected only to shear tractions which are not self-equilibrating the mode I stress intensity factor is nonzero. However, when the crack surfaces are subjected only to normal tractions (which are not necessarily self-equilibrating) the mode II stress intensity factor is necessarily zero. This vanishing of mode II stress intensity factor may be attributed to the axisymmetry of the problem.

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APPENDIX A. SOME SIMPLE INTEGRALS

Using the substitution $\rho = r \sin \theta$ we can show that

$$\frac{2}{\pi}\int_0^r \frac{\rho^2\,\mathrm{d}\rho}{(\rho^2-t^2)\sqrt{r^2-\rho^2}} = \begin{Bmatrix} 1, & t<r \\ 1-\dfrac{t}{\sqrt{t^2-r^2}}, & t>r \end{Bmatrix}. \tag{A1}$$

Let $\beta(r, s)$ be defined by

$$\beta(r,s) = \int_{\max(r,a)}^{s} \frac{t\,\mathrm{d}t}{\sqrt{(s^2-t^2)(t^2-r^2)}}, \quad r \geqslant 0, \quad a \geqslant 0, \tag{A2}$$

where max (r, a) denotes the greater of the two members r and a. The integral may be easily evaluated and we have

$$\beta(r,s) = \begin{Bmatrix} \sin^{-1}\sqrt{(s^2-a^2)/(s^2-r^2)}, & 0 \leqslant r \leqslant a \\ \pi/2, & r>a \end{Bmatrix}. \tag{A3}$$

If $\mathbb{P}(t)$ is defined as in (28), we can write

$$\int_a^\infty \mathbb{P}(t)\,\mathrm{d}t = \int_a^\infty \mathrm{d}t\int_a^t P^*(s)\left[\frac{t}{\sqrt{t^2-s^2}}-1\right]\mathrm{d}s - \int_a^\infty \mathrm{d}t\int_t^\infty P^*(s)\,\mathrm{d}s. \tag{A4}$$

Interchanging the order of integrations and evaluating the resulting inner integrals, we can show that

$$\int_a^\infty \mathbb{P}(t)\,\mathrm{d}t = a\int_a^\infty P^*(s)\,\mathrm{d}s. \tag{A5}$$

APPENDIX B. INTEGRALS REDUCIBLE TO ELLIPTIC INTEGRALS

The elliptic integrals of the second and the first kind are defined by [*cf.* Spanier and Oldham (1987)]

$$E(k) = E\left(\frac{\pi}{2},k\right),\quad E(\phi,k) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta,\quad 0<\phi\leqslant\frac{\pi}{2}, \tag{B1}$$

$$K(k) = F\left(\frac{\pi}{2},k\right),\quad F(\phi,k) = \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}},\quad 0<\phi\leqslant\frac{\pi}{2}. \tag{B2}$$

These integrals can be easily computed for given values of k and ϕ. We give here some integrals which may be written in terms of elliptic integrals:

$$\int_r^\infty \frac{(\sqrt{t^2-\varepsilon^2}-t)\,\mathrm{d}t}{\sqrt{t^2-r^2}} = -rE\left(\frac{\varepsilon}{r}\right)+\frac{(r^2-\varepsilon^2)}{r}K\left(\frac{\varepsilon}{r}\right),\quad r>\varepsilon, \tag{B3}$$

$$\int_c^\infty \frac{(\sqrt{t^2-\varepsilon^2}-t)\,\mathrm{d}t}{\sqrt{t^2-r^2}} = \frac{(c-\sqrt{c^2-\varepsilon^2})\sqrt{c^2-r^2}}{c}-\varepsilon E\left(\sin^{-1}\frac{\varepsilon}{c},\frac{r}{\varepsilon}\right),\quad 0<r<\varepsilon<c, \tag{B4}$$

$$\int_c^\infty \frac{(\sqrt{t^2-\varepsilon^2}-t)\,\mathrm{d}t}{\sqrt{t^2-r^2}} = \frac{(c-\sqrt{c^2-\varepsilon^2})\sqrt{c^2-r^2}}{c}-rE\left(\sin^{-1}\frac{r}{c},\frac{\varepsilon}{r}\right)+\frac{(r^2-\varepsilon^2)}{r}F\left(\sin^{-1}\frac{r}{c},\frac{\varepsilon}{r}\right),\quad \varepsilon<r<c, \tag{B5}$$

$$\int_c^\infty \frac{(\sqrt{t^2-c^2}-t)\,\mathrm{d}t}{\sqrt{t^2-r^2}} = \sqrt{c^2-r^2}-cE\left(\frac{r}{c}\right),\quad 0<r<c, \tag{B6}$$

$$\int_\varepsilon^c \frac{\sqrt{t^2-\varepsilon^2}\,\mathrm{d}t}{\sqrt{t^2-r^2}} = \frac{\sqrt{(c^2-r^2)(c^2-\varepsilon^2)}}{c}-\varepsilon E\left(\frac{r}{\varepsilon}\right)+\varepsilon E\left(\sin^{-1}\frac{\varepsilon}{c},\frac{r}{\varepsilon}\right),\quad 0<r<\varepsilon<c, \tag{B7}$$

$$\int_r^c \frac{\sqrt{t^2-\varepsilon^2}\,\mathrm{d}t}{\sqrt{t^2-r^2}} = \frac{\sqrt{(c^2-r^2)(c^2-\varepsilon^2)}}{c}+r\left[E\left(\sin^{-1}\frac{r}{c},\frac{\varepsilon}{r}\right)-E\left(\frac{\varepsilon}{r}\right)\right]-\frac{(r^2-\varepsilon^2)}{r}\left[F\left(\sin^{-1}\frac{r}{c},\frac{\varepsilon}{r}\right)-K\left(\frac{\varepsilon}{r}\right)\right],\quad \varepsilon<r<c. \tag{B8}$$

We can establish the formulae (B3)–(B8) by carrying out an integration by parts and then using an appropriate substitution. For example, consider the integral

$$I = \int_c^\infty \frac{(\sqrt{t^2-\varepsilon^2}-t)\,\mathrm{d}t}{\sqrt{t^2-r^2}} = \int_c^\infty \frac{t}{\sqrt{t^2-r^2}}\left[\frac{\sqrt{t^2-\varepsilon^2}}{t}-1\right]\mathrm{d}t,\quad 0<r<\varepsilon<c. \tag{B9}$$

Carrying out an integration by parts yields

$$I = \frac{(c-\sqrt{c^2-\varepsilon^2})\sqrt{c^2-r^2}}{c}-\int_c^\infty \frac{\varepsilon^2\sqrt{t^2-r^2}\,\mathrm{d}t}{t^2\sqrt{t^2-\varepsilon^2}},\quad 0<r<\varepsilon<c. \tag{B10}$$

Then, substituting $t = r\operatorname{cosec}\theta$ in (B10) together with the use of relation (B9) yields (B4).